

EQUIVALENCES BETWEEN WEIGHT MODULES VIA $\mathcal{N} = 2$ COSET CONSTRUCTIONS

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ABSTRACT. In this paper we introduce a variant of weight modules for certain conformal vertex superalgebras as an appropriate framework of the $\mathcal{N} = 2$ supersymmetric coset construction. We call them weight-wise admissible modules. Motivated by the work of Feigin-Semikhatov-Tipunin, we give (block-wise) categorical equivalences between the categories of weight-wise admissible modules over $\widehat{\mathfrak{sl}}_2$ and the $\mathcal{N} = 2$ superconformal algebra, induced by the coset construction.

As an application, we obtain some character formulae of modules over the $\mathcal{N} = 2$ superconformal algebra.

1. INTRODUCTION

1.1. Background. The infinitesimal symmetry of 2-dimensional $\mathcal{N} = 2$ supersymmetric conformal field theories is described by the $\mathcal{N} = 2$ superconformal algebra (SCA) introduced in [ABd⁺76], which is a supersymmetric generalization of the Virasoro algebra. There are two inequivalent descriptions of the $\mathcal{N} = 2$ SCA called the untwisted sector and the twisted sector. In this paper we only deal with the untwisted sector. See [Ioh10] and references therein for the details.

In 1980s many physicists extensively studied unitarizable highest weight modules over the $\mathcal{N} = 2$ SCA in order to construct models of rational supersymmetric conformal field theories. Among those modules, the classification of irreducible ones is conjectured by [BFK86], and its mathematically rigorous proof is given by [Ioh10] for the untwisted sector. In its proof the $\mathcal{N} = 2$ supersymmetric coset construction [DVPYZ86] (and [KS89] for general affine Lie algebras) plays an important role to construct some specific unitarizable modules, the so-called minimal unitary series of the $\mathcal{N} = 2$ SCA. In [DVPYZ86] they constructed all the minimal unitary series as direct summands of the tensor products of unitarizable highest weight $\widehat{\mathfrak{sl}}_2$ -modules and the fermionic Fock module. This construction is an analogue to the Goddard-Kent-Olive coset constructions for the case of $\mathcal{N} = 0$ or 1 in [GKO86] and can be reformulated in terms of vertex superalgebras.

For a vertex superalgebra V and its vertex subsuperalgebra W , we denote by $C(V, W)$ the commutant vertex superalgebra of W in V (see Appendix A for the definition). Assume that V is the tensor product of the affine vertex operator algebra (VOA) associated with $\widehat{\mathfrak{sl}}_2$ and the free fermionic vertex operator superalgebra (VOSA), and W is isomorphic to the Heisenberg VOA $M(1)$. Then it is known that the commutant vertex superalgebra $C(V, W)$ is isomorphic to the $\mathcal{N} = 2$ VOSA. By the general theory of vertex superalgebras, there exists a vertex superalgebra

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homomorphism $C(V, W) \otimes W \longrightarrow V$; $A \otimes B \longmapsto A_{(-1)}B$ (see [MN99, Proposition 4.4.1]). In particular, the $\mathcal{N} = 2$ supersymmetric coset construction is regarded as a theory of branching rules with respect to the above homomorphism when both the affine VOA and the corresponding $\mathcal{N} = 2$ VOSA are regular in the sense of [DLM97a].

On the contrary, if these VOSAs are not regular, the complete reducibility of modules fails in general, and the pullback action induced by the above homomorphism is hard to analyze. In [FST98] B. Feigin, A. Semikhatov and I. Tipunin discovered another coset construction, which is given by interchanging the roles of the affine VOA and the $\mathcal{N} = 2$ VOSA. (see also [Ada99]). Namely, they assume that V is the tensor product of the $\mathcal{N} = 2$ VOSA and a certain negative-definite lattice vertex superalgebra, and W is isomorphic to $M(1)$. Then the commutant vertex superalgebra $C(V, W)$ is isomorphic to the affine VOA associated with $\widehat{\mathfrak{sl}}_2$. By using these bidirectional constructions, they investigated an ‘equivalence’ between more general weight modules over $\widehat{\mathfrak{sl}}_2$ and over the $\mathcal{N} = 2$ SCA, which works even for the irregular cases. See [FST98, Section IV] for the details. However, mathematically rigorous descriptions of the ‘equivalence’ and explicit proofs are not given in the literature to the best of our knowledge.

1.2. Main result. In this paper we introduce a natural framework including all the weight modules considered in [FST98]. First, we assume that a conformal vertex superalgebra V contains a vertex subalgebra which is isomorphic to the Heisenberg vertex algebra $M(1)^{\otimes \ell}$ for some ℓ , and fix such a vertex subalgebra. Roughly speaking, we call a weak V -module M a *weight-wise admissible module* if M decomposes into a direct sum of Heisenberg Fock modules as $M(1)^{\otimes \ell}$ -module (see Section 3 for the precise definition). Denote the category of such modules by \mathcal{LW}_V . Note that a weight-wise admissible module is not an admissible V -module in the sense of [DLM97a] in general.

When V is the affine VOA associated with $\widehat{\mathfrak{sl}}_2$ or the $\mathcal{N} = 2$ VOSA, it contains the Heisenberg vertex algebra $M(1)$. We can verify that the category \mathcal{LW}_V decomposes into a direct sum of certain full subcategories which we call *blocks* (Definition 3.5). Now we state the main result: we construct the functors which establish equivalences as \mathbb{C} -linear abelian categories between such blocks of $\widehat{\mathfrak{sl}}_2$ and the $\mathcal{N} = 2$ SCA (Theorem 4.4 and 7.7). In addition, we prove that the functors are “spectral flow equivariant” (Corollary 6.3).

As an application, we obtain new character formulae of weight-wise admissible modules over the $\mathcal{N} = 2$ SCA (Theorem 7.8). More precisely, these formulae are written in terms of the characters of the corresponding $\widehat{\mathfrak{sl}}_2$ -modules by the previous equivalence. In particular, we succeed to reprove the character formulae of the ‘admissible’ $\mathcal{N} = 2$ modules of central charge $c = 3 \left(1 - \frac{2p'}{p}\right)$ ($p \geq 2, p' \geq 1$) in the sense of [FSST99] (see Theorem 7.13).

Note that, if the VOSAs are regular, weight-wise admissible modules over the affine simple VOA and the $\mathcal{N} = 2$ simple VOSA coincide with (ordinary) modules over VOSAs in the sense of [KW94]. In this sense, the notion of weight-wise admissible modules is a natural generalization of that of ordinary modules over VOSAs.

1.3. Further problems. Let us explain some issues which are related with our results. Among non-unitary modules over affine Lie algebras, the Kac-Wakimoto admissible modules at admissible levels are distinguished by their modular invariant properties [KW88]. At the admissible levels, weight-wise admissible modules naturally appear in the study of “fusion products” from a viewpoint of [Gab01] (see also [FM94], [DLM97b]). The Wess-Zumino-Witten models at these levels are believed to relate with the so-called logarithmic conformal field theories, which include non-simple indecomposable modules and allow logarithmic singularities in the corresponding conformal blocks. For instance, the ‘logarithmic’ indecomposable module over $\widehat{\mathfrak{sl}}_2$ constructed in [AM09, Proposition 8.1] gives a specific example of weight-wise admissible modules.

Unfortunately, the simple affine VOAs at the admissible levels are not C_2 -cofinite in general (see [AM95, Proposition 3.4.1] for the \mathfrak{sl}_2 -case). In particular, the characters of some irreducible modules over the admissible affine VOA are not convergent. Contrary to this, by our character formula in Theorem 7.8, we can see that the corresponding characters in the $\mathcal{N} = 2$ side are absolutely convergent in some region (see Remark 3.7 and Proposition 7.1). We expect that the study of the modular property of these functions may be the first step to reveal the relationship between the modular invariance and the fusion rules at the admissible levels, for example, a modification of Verlinde’s formula initiated by [AY92] (see also [Gab01], [CR13]).

1.4. Structure of the paper. We organize this paper as follows. In Section 2, after recalling some relevant facts about vertex superalgebras, we give a brief review of the $\mathcal{N} = 2$ supersymmetric coset constructions. Section 3 is devoted to the definition of weight-wise admissible module over conformal vertex superalgebras with certain additional conditions. We present our main result, the categorical equivalences between blocks, in Section 4 and prove it in Section 5. In Section 6, we discuss the relationship between the functors constructed in Section 4 and the spectral flow automorphisms of $\widehat{\mathfrak{sl}}_2$ and the $\mathcal{N} = 2$ SCA (see Appendix B for the definition). In Section 7, we present some applications of the main result.

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2. PRELIMINARIES

In this section we recall some basic facts about vertex superalgebras (see the book [Kac98] for details) and the vertex superalgebraic formulation of the $\mathcal{N} = 2$ supersymmetric coset constructions.

2.1. Notations. Let \mathfrak{sl}_2 be the simple Lie algebra of type A_1 and

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the standard basis of \mathfrak{sl}_2 . The normalized invariant form $(-|-)$ on \mathfrak{sl}_2 is given by $(X|Y) = \text{tr}(XY)$ for $X, Y \in \mathfrak{sl}_2$. The affinization of \mathfrak{sl}_2 with respect to $(-|-)$ is the Lie algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with the following commutation relations:

$$[X_n, Y_m] = [X, Y]_{n+m} + n(X|Y)K\delta_{n+m,0}, \quad [\widehat{\mathfrak{sl}}_2, K] = \{0\}$$

for $X, Y \in \mathfrak{sl}_2$ and $n, m \in \mathbb{Z}$, where $X_n = X \otimes t^n \in \widehat{\mathfrak{sl}}_2$.

On the other hand, the Neveu-Schwarz sector of the $\mathcal{N} = 2$ superconformal algebra is the Lie superalgebra

$$\mathfrak{ns}_2 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}J_n \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G_r^+ \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G_r^- \oplus \mathbb{C}C$$

whose \mathbb{Z}_2 -grading is given by

$$(\mathfrak{ns}_2)^{\bar{0}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}J_n \oplus \mathbb{C}C, \quad (\mathfrak{ns}_2)^{\bar{1}} = \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G_r^+ \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G_r^-$$

with the following (anti-)commutation relations:

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)C\delta_{n+m,0}, \\ [L_n, J_m] &= -mJ_{n+m}, \quad [L_n, G_r^\pm] = \left(\frac{n}{2} - r\right)G_{n+r}^\pm, \\ [J_n, J_m] &= \frac{n}{3}C\delta_{n+m,0}, \quad [J_n, G_r^\pm] = \pm G_{n+r}^\pm, \\ [G_r^+, G_s^-] &= 2L_{r+s} + (r - s)J_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)C\delta_{r+s,0}, \\ [G_r^+, G_s^+] &= [G_r^-, G_s^-] = 0, \quad [\mathfrak{ns}_2, C] = \{0\}, \end{aligned}$$

for $n, m \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

2.2. VOA associated with $\widehat{\mathfrak{sl}}_2$. For $j, k \in \mathbb{C}$, let $M_{j,k}$ be the Verma module of highest weight $(k - 2j)\Lambda_0 + 2j\Lambda_1$ and $|j, k\rangle$ the highest weight vector of $M_{j,k}$, where Λ_i is the i -th fundamental weight of $\widehat{\mathfrak{sl}}_2$. Throughout this paper, we always assume $k \neq -2$. Then it is well known that the so-called vacuum module $V_k(\mathfrak{sl}_2) := M_{0,k}/U(\widehat{\mathfrak{sl}}_2)F_0|0, k\rangle$ has a VOA structure of central charge $c_k := \frac{3k}{k+2}$. Denote by $\mathbf{1}^k$ the vacuum vector, by $\omega^{\mathfrak{sl}_2}$ the conformal vector and by $L_n^{\mathfrak{sl}_2}$ the linear operator $\omega_{(n+1)}^{\mathfrak{sl}_2}$. To simplify notation, we write \mathbf{H}, \mathbf{E} and \mathbf{F} for $H_{-1}\mathbf{1}^k, E_{-1}\mathbf{1}^k$ and $F_{-1}\mathbf{1}^k$, respectively.

Let $L_k(\mathfrak{sl}_2)$ be the simple quotient vertex algebra of $V_k(\mathfrak{sl}_2)$. Since $L_0(\mathfrak{sl}_2)$ is not a VOA, we assume $k \neq 0$ whenever we consider $L_k(\mathfrak{sl}_2)$. By abuse of notation, we also write \mathbf{X} for the image of $\mathbf{X} \in V_k(\mathfrak{sl}_2)$ in $L_k(\mathfrak{sl}_2)$.

2.3. $\mathcal{N} = 2$ VOSA. Let $\mathfrak{ns}_2 = (\mathfrak{ns}_2)_+ \oplus (\mathfrak{ns}_2)_0 \oplus (\mathfrak{ns}_2)_-$ be the triangular decomposition of \mathfrak{ns}_2 , where

$$\begin{aligned} (\mathfrak{ns}_2)_+ &:= \bigoplus_{n>0} \mathbb{C}L_n \oplus \bigoplus_{n>0} \mathbb{C}J_n \oplus \bigoplus_{r>0} \mathbb{C}G_r^+ \oplus \bigoplus_{r>0} \mathbb{C}G_r^-, \\ (\mathfrak{ns}_2)_- &:= \bigoplus_{n<0} \mathbb{C}L_n \oplus \bigoplus_{n<0} \mathbb{C}J_n \oplus \bigoplus_{r<0} \mathbb{C}G_r^+ \oplus \bigoplus_{r<0} \mathbb{C}G_r^-, \\ (\mathfrak{ns}_2)_0 &:= \mathbb{C}L_0 \oplus \mathbb{C}J_0 \oplus \mathbb{C}C, \end{aligned}$$

and set $(\mathfrak{ns}_2)_{\geq 0} := (\mathfrak{ns}_2)_+ \oplus (\mathfrak{ns}_2)_0$.

For $h, j, c \in \mathbb{C}$, let $\mathbb{C}_{h,j,c}$ be the 1-dimensional $(\mathfrak{ns}_2)_{\geq 0}$ -module defined by $(\mathfrak{ns}_2)_{+.1} := \{0\}$, $L_{0.1} := h$, $J_{0.1} := j$ and $C.1 := c$. We call the induced module $\mathcal{M}_{h,j,c} := \text{Ind}_{(\mathfrak{ns}_2)_{\geq 0}}^{\mathfrak{ns}_2} \mathbb{C}_{h,j,c}$ the *Verma module* of \mathfrak{ns}_2 . Let us denote by $|h, j, c\rangle^{\mathfrak{ns}_2}$ the canonical generator $1 \otimes 1 \in \mathcal{M}_{h,j,c}$ and by $\mathcal{L}_{h,j,c}$ the irreducible quotient of $\mathcal{M}_{h,j,c}$.

When $h = \pm \frac{j}{2}$, the quotient modules

$$\mathcal{M}_{j,c}^{\pm} := \mathcal{M}_{\pm \frac{j}{2}, j, c} / U(\mathfrak{ns}_2) G_{-\frac{j}{2}}^{\pm} |\pm \frac{j}{2}, j, c\rangle^{\mathfrak{ns}_2}$$

are called the *chiral Verma module* and the *anti-chiral Verma module*. If $h = j = 0$, according to [Ada99, Proposition 1.1], the quotient module

$$V_c(\mathfrak{ns}_2) := \mathcal{M}_{0,0,c} / (U(\mathfrak{ns}_2) G_{-\frac{1}{2}}^+ |0, 0, c\rangle^{\mathfrak{ns}_2} + U(\mathfrak{ns}_2) G_{-\frac{1}{2}}^- |0, 0, c\rangle^{\mathfrak{ns}_2})$$

has a VOSA structure of central charge c , where the vacuum $\mathbf{1}^c$ is given by the image of $|0, 0, c\rangle^{\mathfrak{ns}_2}$ in $V_c(\mathfrak{ns}_2)$ and the conformal vector $\omega^{\mathfrak{ns}_2}$ is given by $L_{-2}\mathbf{1}^c$. For simplicity of notation, we write \mathbf{J} and \mathbf{G}^{\pm} for $J_{-1}\mathbf{1}^c$ and $G_{-\frac{3}{2}}^{\pm}\mathbf{1}^c$, respectively.

Let $L_c(\mathfrak{ns}_2)$ be the simple quotient vertex superalgebra of $V_c(\mathfrak{ns}_2)$. Note that $L_c(\mathfrak{ns}_2) = \mathcal{L}_{0,0,c}$ as an \mathfrak{ns}_2 -module. Similarly to the affine case, we write \mathbf{X} for the image of $\mathbf{X} \in V_c(\mathfrak{ns}_2)$ in $L_c(\mathfrak{ns}_2)$.

2.4. Heisenberg VOAs and lattice vertex superalgebras. In order to formulate the $\mathcal{N} = 2$ supersymmetric coset constructions, we have to introduce the Heisenberg VOAs \mathcal{F}^{\pm} and the lattice conformal vertex superalgebra V^{\pm} associated with certain non-degenerate integral lattices. See [Kac98] for the details.

Let $Q^{\pm} = \mathbb{Z}\alpha^{\pm}$ be the integral lattices with the \mathbb{Z} -bilinear forms defined by $(\alpha^{\pm}|\alpha^{\pm}) = \pm 1$, respectively. Let $\hat{\mathfrak{t}}^{\pm} = Q^{\pm} \otimes_{\mathbb{Z}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K^{\pm}$ be the corresponding affinizations, *i.e.* the commutation relations are given by

$$[\alpha_n^{\pm}, \alpha_m^{\pm}] = \pm n K^{\pm} \delta_{n+m, 0}, \quad [\hat{\mathfrak{t}}^{\pm}, K^{\pm}] = \{0\},$$

where $\alpha_n^{\pm} := \alpha^{\pm} \otimes t^n$.

For $\lambda \in \mathbb{C}$, let $\mathcal{F}_{\lambda}^{\pm}$ be the Heisenberg Fock module of charge λ and level 1. Denote by $|\lambda\rangle^{\pm}$ the highest weight vector of $\mathcal{F}_{\lambda}^{\pm}$. The Heisenberg VOA \mathcal{F}^{\pm} is the $\hat{\mathfrak{t}}^{\pm}$ -module \mathcal{F}_0^{\pm} with a VOA structure of central charge 1, where the vacuum vector is $|0\rangle^{\pm}$ and the conformal vector is $\omega^{\pm} := \pm \frac{1}{2}(\alpha_{-1}^{\pm})^2 |0\rangle^{\pm}$. We write α^{\pm} for $\alpha_{-1}^{\pm} |0\rangle^{\pm} \in \mathcal{F}^{\pm}$ and L_n^{\pm} for the linear operator $\omega_{(n+1)}^{\pm}$.

Remark 2.1. The VOAs \mathcal{F}^+ , \mathcal{F}^- and $M(1)$ are all isomorphic.

We also recall the simple lattice vertex superalgebras associated with non-degenerate integral lattices Q^{\pm} . Let $\mathbb{C}[Q^{\pm}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{n\alpha^{\pm}}$ be the group algebra of Q^{\pm} over \mathbb{C} . It is known that there exists a simple vertex superalgebra structure on the \mathbb{Z}_2 -graded vector space $V_{Q^{\pm}} := \mathbb{C}[Q^{\pm}] \otimes_{\mathbb{C}} \text{Sym}(\hat{\mathfrak{t}}_{<0}^{\pm})$ with the vacuum vector $\mathbf{1}^{\pm} := e^0 \otimes 1$. Here the \mathbb{Z}_2 -grading on $V_{Q^{\pm}}$ is defined by

$$(V_{Q^{\pm}})^{\bar{i}} := \bigoplus_{(\beta|\beta) \in \{i\} + 2\mathbb{Z}} \mathbb{C}e^{\beta} \otimes_{\mathbb{C}} \mathbb{C}[\alpha_{-n}^{\pm} | n > 0]$$

for $i \in \{0, 1\}$. See [Kac98, Theorem 5.5] for the details.

We take $\omega^{Q^{\pm}} := e^0 \otimes (\pm \frac{1}{2}(\alpha_{-1}^{\pm})^2)$ as the conformal vector of $V_{Q^{\pm}}$ of central charge 1. Denote by V^{\pm} the corresponding conformal vertex superalgebra and by $L_n^{Q^{\pm}}$ the linear operator $\omega_{(n+1)}^{Q^{\pm}}$. Note that V^- is not a VOSA. In what follows, we abbreviate $e^0 \otimes u$ as u and $e^{n\alpha^{\pm}} \otimes 1$ as $e^{n\alpha^{\pm}}$, where $u \in \mathbb{C}[\alpha_{-n}^{\pm} | n > 0]$.

2.5. Formulation of $\mathcal{N} = 2$ coset constructions. We give a review of the vertex superalgebraic formulation of the $\mathcal{N} = 2$ supersymmetric coset construction [DVPYZ86] and its “inverse” coset construction [FST98]. The following reformulation is due to [Ada99]. In what follows, we always take $\kappa \in \mathbb{C}^\times$ and assume $k = -2 + \frac{2}{\kappa^2}$.

Fact 2.2. (1) There exists a unique homomorphism of conformal vertex superalgebras $\iota_+ : V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \rightarrow V_k(\mathfrak{sl}_2) \otimes V^+$ such that

$$\begin{aligned} \mathbf{G}^+ \otimes |0\rangle^+ &\mapsto \kappa \mathbf{E} \otimes e^{\alpha^+}, \\ \mathbf{G}^- \otimes |0\rangle^+ &\mapsto \kappa \mathbf{F} \otimes e^{-\alpha^+}, \\ \mathbf{J} \otimes |0\rangle^+ &\mapsto \frac{\kappa^2}{2} \mathbf{H} \otimes \mathbf{1}^+ + \frac{c_k}{3} \mathbf{1}^k \otimes \alpha_{-1}^+, \\ \mathbf{1}^{c_k} \otimes \alpha^+ &\mapsto \kappa \left(\frac{1}{2} \mathbf{H} \otimes \mathbf{1}^+ - \mathbf{1}^k \otimes \alpha_{-1}^+ \right). \end{aligned}$$

(2) There exists a unique homomorphism of conformal vertex superalgebras $\iota_- : V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^- \rightarrow V_{c_k}(\mathfrak{ns}_2) \otimes V^-$ such that

$$\begin{aligned} \mathbf{E} \otimes |0\rangle^- &\mapsto \kappa^{-1} \mathbf{G}^+ \otimes e^{-\alpha^-}, \\ \mathbf{F} \otimes |0\rangle^- &\mapsto \kappa^{-1} \mathbf{G}^- \otimes e^{\alpha^-}, \\ \mathbf{H} \otimes |0\rangle^- &\mapsto \frac{2}{\kappa^2} \mathbf{J} \otimes \mathbf{1}^- - k \mathbf{1}^{c_k} \otimes \alpha_{-1}^-, \\ \mathbf{1}^k \otimes \alpha^- &\mapsto \kappa^{-1} (\mathbf{J} \otimes \mathbf{1}^- - \mathbf{1}^{c_k} \otimes \alpha_{-1}^-). \end{aligned}$$

In what follows, we always abbreviate $A \otimes \mathbf{1}_W$ and $\mathbf{1}_V \otimes B$ in a tensor product vertex superalgebra $V \otimes W$ as A and B , respectively.

Let σ be the symmetric braiding of the category of \mathbb{Z}_2 -graded vector spaces. Then we define $\iota_{\mathfrak{sl}_2} := (\iota_+)_{12} \circ (\iota_-)_{13}$ and $\iota_{\mathfrak{ns}_2} := (\iota_-)_{13} \circ (\iota_+)_{12}$, where $(\iota_+)_{12} := \iota_+ \otimes \text{id}$ and $(\iota_-)_{13} := (\text{id} \otimes \sigma) \circ (\iota_- \otimes \text{id}) \circ (\text{id} \otimes \sigma)$.

Corollary 2.3. The mappings $\iota_{\mathfrak{sl}_2} : V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \rightarrow V_k(\mathfrak{sl}_2) \otimes V^+ \otimes V^-$ and $\iota_{\mathfrak{ns}_2} : V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \rightarrow V_{c_k}(\mathfrak{ns}_2) \otimes V^+ \otimes V^-$ are morphisms of conformal vertex superalgebras.

3. WEIGHT-WISE ADMISSIBLE MODULES

In this section we introduce the notion of weight-wise admissible modules. The category of such modules is a variant of the “category \mathcal{O} ” for a VOA firstly introduced by [DLM97a, Section 2].

3.1. General notations. Let $(V, Y, \mathbf{1}, \omega)$ be a conformal vertex superalgebra. Assume that there exists a finite linearly independent subset $S = \{a^i\}_{i=1}^\ell$ of $V_1 \cap V^{\bar{0}}$ such that S generates the purely even vertex subalgebra $\langle S \rangle$ of V which is isomorphic to the Heisenberg vertex algebra $M(1)^{\otimes \ell}$. We define the action of the abelian Lie algebra $\mathfrak{h} := \text{span}_{\mathbb{C}} S$ on M by $a^i \mapsto a_{(0)}^i$. In what follows, we fix the above pair (V, S) .

Definition 3.1. A weak V -module M is a *weight-wise admissible* (V, S) -module if

- (1) We have the decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M(\lambda)$ as an \mathfrak{h} -module, where $M(\lambda) := \{m \in M \mid h.m = \langle \lambda, h \rangle m \text{ for any } h \in \mathfrak{h}\}$.
- (2) For any $\lambda \in \mathfrak{h}^*$, we have the generalized eigenspace decomposition $M(\lambda) = \bigoplus_{h \in \mathbb{C}} M(h, \lambda)$ with respect to L_0 and $\dim M(h, \lambda) \leq \infty$.
- (3) The set $\{\text{Re}(h) \mid M(h, \lambda) \neq \{0\}\}$ is lower bounded for any $\lambda \in \mathfrak{h}^*$.

Definition 3.2. For a weight-wise admissible (V, S) -module M , we define $P(M) := \{(h, \lambda) \in \mathbb{C} \times \mathfrak{h}^* \mid M(h, \lambda) \neq \{0\}\}$.

In what follows, we fix the isomorphism $\mathfrak{h}^* \cong \mathbb{C}^\ell$ by the dual basis of S .

Definition 3.3. Suppose that $P(M)$ is discrete. Then we define the formal character of M by

$$\text{ch}(M)(q, x_1, \dots, x_l) := \sum_{(h, \lambda) \in P(M)} (\dim M(h, \lambda)) q^h \mathbf{x}^\lambda,$$

where $\mathbf{x}^\lambda := x_1^{\lambda_1} \cdots x_l^{\lambda_l}$ for $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{C}^l$.

The following lemma describes the $\langle S \rangle$ -module structure of a weight-wise admissible (V, S) -module.

Lemma 3.4. Let M be a weight-wise admissible (V, S) -module and $M(1, \lambda) = \bigotimes_{i=1}^\ell M(1, \lambda_i)$ the Heisenberg Fock module over $M(1)^{\otimes \ell} (\cong \langle S \rangle)$ for $\lambda = (\lambda_1, \dots, \lambda_l)$. Then the linear mapping

$$\bigoplus_{\lambda \in \mathbb{C}^\ell} \text{Hom}_{M(1)^{\otimes \ell}}(M(1, \lambda), M) \otimes_{\mathbb{C}} M(1, \lambda) \rightarrow M; f \otimes v \mapsto f(v)$$

is an isomorphism of a weak $M(1)^{\otimes \ell}$ -module.

Proof. The injectivity of the mapping follows from the weight space decomposition and the irreducibility of $M(1, \lambda)$. By the condition (3) in Definition 3.1, $M(\lambda)$ satisfies the condition \mathfrak{C}_1 in [FLM89, Section 1.7] as a module over the Heisenberg Lie algebra associated with $M(1)^{\otimes \ell}$. Then the surjectivity follows from [FLM89, Theorem 1.7.3]. \square

Denote by \mathcal{LW}_V the full subcategory of $V\text{-Mod}$ whose objects are weight-wise admissible V -modules. Since weight-wise admissible V -modules are closed under finite sums, kernels and cokernels, the category \mathcal{LW}_V is abelian.

Definition 3.5. Assume that (V, Y) is a weight-wise admissible (V, S) -module. Let Q_S be the additive subgroup of $\mathbb{C} \times \mathfrak{h}^*$ generated by $P(V)$. For $h \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*$, the $\overline{(h, \lambda)}$ -block $\mathcal{LW}_V^{\overline{(h, \lambda)}}$ is the full subcategory of \mathcal{LW}_V whose objects satisfy $P(M) \subset \overline{(h, \lambda)} := \{(h, \lambda)\} + Q_S$.

3.2. Our setting. In what follows, we denote by $V_{\mathfrak{sl}_2}$ the VOA $V_k(\mathfrak{sl}_2)$ or $L_k(\mathfrak{sl}_2)$, and by $V_{\mathfrak{ns}_2}$ the VOSA $V_{c_k}(\mathfrak{ns}_2)$ or $L_{c_k}(\mathfrak{ns}_2)$. Throughout this paper, we fix the following vectors:

$$S(V_{\mathfrak{sl}_2}) := \left\{ \frac{1}{2} \mathbf{H} \right\}, \quad S(V_{\mathfrak{ns}_2}) := \{ \mathbf{J} \}, \quad S(\mathcal{F}^\pm) := \{ \alpha^\pm \}, \quad S(V^\pm) := \{ \alpha_{-1}^\pm \}.$$

Let V and W be one of the above conformal vertex superalgebras. For the tensor vertex superalgebra $V \otimes W$, we put $S(V \otimes W) := S(V) \sqcup S(W)$ via the natural embeddings $V \rightarrow V \otimes W$ and $W \rightarrow V \otimes W$. It is easy to verify that the adjoint module (V, Y) is a weight-wise admissible $(V, S(V))$ -module. By easy computations, we obtain

$$Q_{S(V_{\mathfrak{sl}_2})} = \mathbb{Z}^2, \quad Q_{S(V_{\mathfrak{ns}_2})} = \left\{ \left(a + \frac{b}{2}, b \right) \mid a, b \in \mathbb{Z} \right\} \cong \mathbb{Z}^2, \\ Q_{S(\mathcal{F}^\pm)} = \mathbb{Z} \times \{0\} \cong \mathbb{Z}, \quad Q_{S(V^\pm)} = \left\{ \left(a + \frac{b}{2}, b \right) \mid a, b \in \mathbb{Z} \right\} \cong \mathbb{Z}^2.$$

3.3. Examples. In this subsection we present some concrete examples.

3.3.1. Highest weight modules over $\widehat{\mathfrak{sl}}_2$ and \mathfrak{ns}_2 . The Verma module $M_{j,k}$ and its quotient modules lie in the block $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(\Delta_j, j)}$, where $\Delta_j := \frac{j(j+1)}{k+2}$. On the other hand, the Verma module $\mathcal{M}_{h,j,c}$ and its quotients lie in the block $\mathcal{LW}_{V_c(\mathfrak{ns}_2)}^{(h,j)}$.

3.3.2. Relaxed highest weight modules over $\widehat{\mathfrak{sl}}_2$. Since the block $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j)}$ contains no highest weight modules for generic $h, j \in \mathbb{C}$, we introduce a generalization of usual highest weight modules as the appropriate counterpart of the highest weight \mathfrak{ns}_2 -modules.

Let $U_0 := \{u \in U(\mathfrak{sl}_2) \mid [H, u] = 0\}$ be the subalgebra of $U(\mathfrak{sl}_2)$. It is clear that H and the quadratic Casimir element $\Omega \in U(\mathfrak{sl}_2)$ freely generate U_0 as a unital commutative \mathbb{C} -algebra. For $h, j \in \mathbb{C}$, let $\mathbb{C}_{h,j}$ be the 1-dimensional U_0 -module defined by $\Omega \cdot 1 := 2(k+2)h$ and $H \cdot 1 := 2j$. We define the action of $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ on the induced \mathfrak{sl}_2 -module $\text{Ind}_{U_0}^{U(\mathfrak{sl}_2)} \mathbb{C}_{h,j,k}$ by $X_n \mapsto X\delta_{n,0}$, $K \mapsto k \text{id}$ for $X \in \mathfrak{sl}_2$ and $n \geq 0$. Denote this $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ -module by $\bar{R}_{h,j,k}$. Then, we define the relaxed Verma module (cf. [FST98]) as the induced module $R_{h,j,k} := \text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{sl}}_2} \bar{R}_{h,j,k}$. We write $|h, j, k\rangle$ for the canonical generator of $R_{h,j,k}$.

By an easy computation, we have $L_0^{\mathfrak{sl}_2} |h, j, k\rangle = h |h, j, k\rangle$. Hence $R_{h,j,k}$ lies in $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j)}$. Since the sum of all proper submodules of $R_{h,j,k}$ which intersect trivially with $\mathbb{C} |h, j, k\rangle$ gives the maximum proper submodule, there exists a unique irreducible quotient $L_{h,j,k}$ of $R_{h,j,k}$.

In [Fut96], V. Futorny introduces a generalization of Verma modules over the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2 \rtimes \mathbb{C}D$. It is clear that the modules are isomorphic to relaxed Verma modules with spectral flow twists (see Appendix B) as $\widehat{\mathfrak{sl}}_2$ -modules. The next fact immediately follows from [Fut96, Theorem 6.3], the classification of irreducible weight modules over $\widehat{\mathfrak{sl}}_2 \rtimes \mathbb{C}D$.

Fact 3.6. Assume $k \neq 0$ and let L be a simple object in $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}$. Then, there exist $h, j \in \mathbb{C}$ and $\theta \in \mathbb{Z}$ such that $L \cong L_{h,j,k}^\theta$.

Remark 3.7. By some computations, we have

$$\text{ch}(R_{h,j,k})(q, x) = q^h x^j \sum_{n \in \mathbb{Z}} \frac{x^n}{(q; q)_\infty^3},$$

where $(a; q)_\infty := \prod_{i \geq 0} (1 - aq^i)$. Note that the formal series $\text{ch}(\mathcal{M}_{h,j,c})(q, x)$ is absolutely convergent as a function in two variables q and x in the region $\mathbb{A} := \left\{ (x, q) \in \mathbb{C}^2 \mid 0 < |q| < 1, |q|^{\frac{1}{2}} < |x| < |q|^{-\frac{1}{2}} \right\}$, while $\text{ch}(R_{h,j,k})(q, x)$ has no convergent region.

4. EQUIVALENCE BETWEEN MODULE CATEGORIES

In this section we construct functors which establish categorical equivalences, and state the main result. Throughout this section, we fix $h, j \in \mathbb{C}$.

4.1. Functors Ω_j^\pm . In this subsection we introduce functors between $V_k(\mathfrak{sl}_2)\text{-Mod}$ and $V_{c_k}(\mathfrak{ns}_2)\text{-Mod}$. The restrictions of these functors give categorical equivalences between blocks of the category of weight-wise admissible modules.

Definition 4.1. We define the functors

$$\begin{aligned}\Omega_j^+ &:= \text{Hom}_{\mathcal{F}^+}(\mathcal{F}_{\kappa j}^+, \iota_+^*(- \otimes V^+)) : V_k(\mathfrak{sl}_2)\text{-Mod} \rightarrow V_{c_k}(\mathfrak{ns}_2)\text{-Mod}, \\ \Omega_j^- &:= \text{Hom}_{\mathcal{F}^-}(\mathcal{F}_{\kappa j}^-, \iota_-^*(- \otimes V^-)) : V_{c_k}(\mathfrak{ns}_2)\text{-Mod} \rightarrow V_k(\mathfrak{sl}_2)\text{-Mod}.\end{aligned}$$

In what follows, we always suppose

$$(4.1) \quad M \in \text{Obj}\left(\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}}\right) \text{ and } N \in \text{Obj}\left(\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}}\right).$$

Lemma 4.2. We have

$$\Omega_j^+(M) \in \text{Obj}\left(\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}}\right) \text{ and } \Omega_j^-(N) \in \text{Obj}\left(\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}}\right).$$

Proof. Since ι_\pm preserve the subspaces $\mathfrak{h} = \text{span}_{\mathbb{C}} S$ of the corresponding vertex superalgebras, the conditions in Definition 3.1 for $\Omega_j^+(M)$ and $\Omega_j^-(N)$ inherit from those for M , N and V^\pm . By easy computations, it follows that these modules lie in the above blocks. \square

To simplify notation, we set

$$\mathcal{F}_{(n,m)}^+ := \mathcal{F}_{\kappa(j+\frac{k}{2}n-m)}^+, \quad \mathcal{F}_{(n,m)}^- := \mathcal{F}_{\kappa(j+\frac{k}{2}n+\frac{k+2}{2}m)}^-$$

for $n, m \in \mathbb{Z}$. The functors Ω_j^\pm are motivated by the following isomorphisms.

Lemma 4.3. (1) The linear mapping

$$(4.2) \quad \bigoplus_{n \in \mathbb{Z}} \Omega_{j-n}^+(M) \otimes \mathcal{F}_{(0,n)}^+ \xrightarrow{\cong} \iota_+^*(M \otimes V^+); f \otimes v \mapsto f(v)$$

is a $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module isomorphism.

(2) The linear mapping

$$(4.3) \quad \bigoplus_{m \in \mathbb{Z}} \Omega_{j+\frac{k+2}{2}m}^-(N) \otimes \mathcal{F}_{(0,m)}^- \xrightarrow{\cong} \iota_-^*(N \otimes V^-); g \otimes w \mapsto g(w)$$

is a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^-$ -module isomorphism.

Since the proof is similar to that of Lemma 3.4, we omit it.

4.2. Statement of main result. In this subsection we state the main result in this paper. We prove it in the next section.

Denote by $\Omega_j^+|$ and $\Omega_j^-|$ the restrictions of the functors Ω_j^+ and Ω_j^- to the full subcategories $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}}$ and $\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}}$, respectively. Then, our result is as follows:

Theorem 4.4. The two functors

$$\Omega_j^+| : \mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}} \rightarrow \mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}}, \quad \Omega_j^-| : \mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}} \rightarrow \mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}}$$

are mutually quasi-inverse to each other. In particular, these functors give categorical equivalences as \mathbb{C} -linear abelian categories.

5. PROOF OF MAIN RESULT

In this section we give the proof of Theorem 4.4 by steps. Throughout this section, we fix $h, j \in \mathbb{C}$ and suppose that

$$M \in \text{Obj} \left(\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j)}} \right) \quad \text{and} \quad N \in \text{Obj} \left(\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}} \right).$$

5.1. Branching rules with respect to $\iota_{\mathfrak{sl}_2}^*$ and $\iota_{\mathfrak{ns}_2}^*$.

Lemma 5.1. (1) As a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module, we have

$$\bigoplus_{n,m \in \mathbb{Z}} \Omega_{j+\frac{k}{2}n+\frac{k+2}{2}m}^- \circ \Omega_{j-n}^+(M) \otimes \mathcal{F}_{(0,n)}^+ \otimes \mathcal{F}_{(n,m)}^- \xrightarrow{\cong} \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-).$$

(2) As a $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module, we have

$$\bigoplus_{m \in \mathbb{Z}} \Omega_{j+\frac{k}{2}m-n}^+ \circ \Omega_{j+\frac{k+2}{2}m}^-(N) \otimes \mathcal{F}_{(m,n)}^+ \otimes \mathcal{F}_{(0,m)}^- \xrightarrow{\cong} \iota_{\mathfrak{ns}_2}^*(N \otimes V^+ \otimes V^-).$$

Proof. Since $\iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-) = (\iota_-)^*_{13} \circ (\iota_+)^*_{12}(M \otimes V^+ \otimes V^-)$, there exists a natural $V_{c_k}(\mathfrak{ns}_2)$ -module isomorphism

$$\Omega_{j+\frac{k}{2}n+\frac{k+2}{2}m}^- \circ \Omega_{j-n}^+(M) \cong \text{Hom}_{\mathcal{F}^+ \otimes \mathcal{F}^-}(\mathcal{F}_{(0,n)}^+ \otimes \mathcal{F}_{(n,m)}^-, \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-)).$$

Through the natural isomorphism, we define a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module homomorphism

$$\bigoplus_{n,m \in \mathbb{Z}} \Omega_{j+\frac{k}{2}n+\frac{k+2}{2}m}^- \circ \Omega_{j-n}^+(M) \otimes \mathcal{F}_{(0,n)}^+ \otimes \mathcal{F}_{(n,m)}^- \rightarrow \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-)$$

by $f \otimes v^+ \otimes v^- \mapsto f(v^+ \otimes v^-)$. The bijectivity of this mapping is proved in a similar way as Lemma 3.4. The proof of (2) is same as that of (1). \square

5.2. Calculation of formal characters. Let $\theta \in \mathbb{Z}$. We use the following notations:

$$\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j;\theta)}} := \mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h+j\theta+\frac{k\theta^2}{4}, j+\frac{k\theta}{2})}}, \quad \mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h,j;\theta)}} := \mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h-\frac{(j-\theta)^2}{k+2}, \frac{2(j-\theta)}{k+2})}}.$$

Lemma 5.2. For $\theta \in \mathbb{Z}$, the twisted modules M^θ and N^θ (see Appendix B for the definition) lie in $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{\overline{(h,j;\theta)}}$ and $\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{\overline{(h,j;\theta)}}$, respectively. Moreover, we have $\text{ch}(M^\theta)(q, x) = q^{\frac{k\theta^2}{4}} x^{\frac{k\theta}{2}} \text{ch}(M)(q, xq^\theta)$ and $\text{ch}(N^\theta)(q, x) = q^{\frac{k\theta^2}{2(k+2)}} x^{\frac{k\theta}{k+2}} \text{ch}(N)(q, xq^\theta)$.

Proof. As a corollary of Lemma B.4, we see that the linear operator $(\omega^{\mathfrak{sl}_2})_{(1)}^{M^\theta}$ coincides with $L_0^{\mathfrak{sl}_2} + \frac{\theta}{2}H_0 + \frac{\theta^2}{4}K$ as an element of $\text{End}(M)$. For $(h', j') \in \mathbb{C}^2$, we put $(\tilde{h}', \tilde{j}') := (h' + j'\theta + \frac{k\theta^2}{4}, j' + \frac{k\theta}{2}) \in \mathbb{C}^2$. Since $(\tilde{h}', \tilde{j}') = (\tilde{h}'', \tilde{j}'')$ holds if and only if $(h', j') = (h'', j'')$, we have $\dim M_{h', j'} = \dim(M^\theta)_{\tilde{h}', \tilde{j}'}$. Hence M^θ inherits the conditions of weight-wise admissibility and the former equality holds.

Since we have $\dim N_{h, \lambda} = \dim(N^\theta)_{h+\theta\lambda+\frac{k\theta^2}{2(k+2)}, \lambda+\frac{k\theta}{k+2}}$ by Lemma B.4, the statements for N^θ are proved in the same way. \square

As a corollary, we have the following categorical isomorphisms, which play important roles in the next section.

Corollary 5.3. The restrictions of the functors $\Delta_{\mathfrak{sl}_2}^\theta$ and $\Delta_{\mathfrak{ns}_2}^\theta$ to the full subcategories $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j;0)}$ and $\mathcal{LW}_{V_c(\mathfrak{ns}_2)}^{(h,j;0)}$ give categorical isomorphisms $\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j;0)} \xrightarrow{\cong} \mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j;\theta)}$ and $\mathcal{LW}_{V_c(\mathfrak{ns}_2)}^{(h,j;0)} \xrightarrow{\cong} \mathcal{LW}_{V_c(\mathfrak{ns}_2)}^{(h,j;\theta)}$, respectively.

Lemma 5.4. (1) As a formal character of $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module, we have

$$\text{ch} \left(\bigoplus_{n,m \in \mathbb{Z}} M^{\theta+n+m} \otimes \mathcal{F}_{(\theta,n)}^+ \otimes \mathcal{F}_{(\theta+n,m)}^- \right) = \text{ch} \left(\iota_{\mathfrak{sl}_2}^* (M^\theta \otimes V^+ \otimes V^-) \right).$$

(2) As a formal character of $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module, we have

$$\text{ch} \left(\bigoplus_{n,m \in \mathbb{Z}} N^{\theta+n+m} \otimes \mathcal{F}_{(\theta+m,n)}^+ \otimes \mathcal{F}_{(\theta,m)}^- \right) = \text{ch} \left(\iota_{\mathfrak{ns}_2}^* (N^\theta \otimes V^+ \otimes V^-) \right).$$

Proof. Since M and N lie in blocks, there exist $f(q, x), g(q, x) \in \mathbb{Z}_{\geq 0}[[q^{\pm 1}, x^{\pm 1}]]$ such that $\text{ch}(M)(q, x) = q^h x^j f(q, x)$ and $\text{ch}(N)(q, x) = q^{h-\frac{j^2}{k+2}} x^{\frac{2j}{k+2}} g(q, x)$.

(1) For simplicity, we replace the parameters by $x := x_1, y := (x_2)^\kappa$ and $z := (x_3)^\kappa$ for weight-wise admissible $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -modules. Then, the left- and right-hand sides (LHS and RHS for short) are calculated as

$$\begin{aligned} (\text{LHS}) &= \frac{q^{h+j\theta+\frac{k\theta^2}{4}} (xyz)^{j+\frac{k\theta}{2}}}{(q; q)_\infty^2} \sum_{n,m \in \mathbb{Z}} q^{\Delta_{n,m}} (xz)^{\frac{k}{2}(n+m)} y^{-n} z^m f(q, xq^{\theta+n+m}), \\ (\text{RHS}) &= \text{ch} \left(M^\theta \otimes V^+ \otimes V^- \right) (q, xyz, y^{-1}(xz)^{\frac{k}{2}}, z^{-1}(xz)^{-\frac{k}{2}}) \\ &= \frac{q^{h+j\theta+\frac{k\theta^2}{4}} (xyz)^{j+\frac{k\theta}{2}}}{(q; q)_\infty^2} \sum_{n,m \in \mathbb{Z}} q^{\Delta_{n,m}} (xz)^{\frac{k}{2}(n+m)} y^{-n} z^m f(q, xyzq^\theta) \end{aligned}$$

where $\Delta_{n,m} := \frac{(n+m)(n-m)}{2}$. So we can reduce them to

$$\begin{aligned} (\text{LHS})' &:= \sum_{n,m \in \mathbb{Z}} q^{\Delta_{n,m}} (xz)^{\frac{k}{2}(n+m)} y^{-n} z^m f(q, xq^{\theta+n+m}), \\ (\text{RHS})' &:= \sum_{n,m \in \mathbb{Z}} q^{\Delta_{n,m}} (xz)^{\frac{k}{2}(n+m)} y^{-n} z^m f(q, xyzq^\theta). \end{aligned}$$

Here we put $\ell := n + m, A := xq^{\theta+\ell}, B := xyzq^\theta$ and $C := y^{-1}(xz)^{\frac{k}{2}}$, then

$$\begin{aligned} (\text{LHS})' &= \sum_{\ell \in \mathbb{Z}} q^{\frac{\ell^2}{2}} C^\ell \sum_{m \in \mathbb{Z}} \left(\frac{B}{A} \right)^m f(q, A), \\ (\text{RHS})' &= \sum_{\ell \in \mathbb{Z}} q^{\frac{\ell^2}{2}} C^\ell \sum_{m \in \mathbb{Z}} \left(\frac{B}{A} \right)^m f(q, B). \end{aligned}$$

Since A and B are independent of $m \in \mathbb{Z}$, and $f(q, X)$ is an element of $\mathbb{Z}_{\geq 0}[[q^{\pm 1}, X^{\pm 1}]]$, we have an equality

$$\sum_{m \in \mathbb{Z}} \left(\frac{B}{A} \right)^m f(q, A) = \sum_{m \in \mathbb{Z}} \left(\frac{B}{A} \right)^m f(q, B)$$

in $\mathbb{Z}_{\geq 0}[[q^{\pm 1}, x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]]$. We thus get $(\text{LHS}) = (\text{RHS})$.

(2) The proof is similar to (1) and we omit it. \square

5.3. Twisted embedding. In this subsection we give more explicit ‘branching’ rules than Lemma 5.1, which are generalizations of the results in [FST98].

By the assumptions on M and N , we have the eigenspace decompositions $M = \bigoplus_{a \in \mathbb{Z}} M(j+a)$ and $N = \bigoplus_{a \in \mathbb{Z}} N(\kappa^2 j + a)$. From now on, we always denote the decompositions for $v \in M$ and $w \in N$ by $v = \sum_a v_a \in \bigoplus_{a \in \mathbb{Z}} M(j+a)$ and $w = \sum_a w_a \in \bigoplus_{a \in \mathbb{Z}} N(\kappa^2 j + a)$, respectively.

First, we introduce the following two operators:

$$\mathcal{H} := \sum_{n>0} \frac{1}{2} H_n \otimes \left(\frac{\alpha_{-n}^+}{n} \otimes \text{id}_{V^-} - \text{id}_{V^+} \otimes \frac{\alpha_{-n}^-}{n} \right) \in \text{End}(M \otimes V^+ \otimes V^-),$$

$$\mathcal{J} := \sum_{n>0} J_n \otimes \left(\frac{\alpha_{-n}^+}{n} \otimes \text{id}_{V^-} - \text{id}_{V^+} \otimes \frac{\alpha_{-n}^-}{n} \right) \in \text{End}(N \otimes V^+ \otimes V^-).$$

Lemma 5.5. The infinite sums $e^{\mathcal{H}} := \exp(\mathcal{H})$ and $e^{\mathcal{J}} := \exp(\mathcal{J})$ give well-defined operators on $M \otimes V^+ \otimes V^-$ and $N \otimes V^+ \otimes V^-$, respectively.

Proof. By the condition (3) in Definition 3.1, we have $\mathcal{H}^m(M(j+a) \otimes V^+ \otimes V^-) = \{0\}$ for any $a \in \mathbb{Z}$ and $m \gg 0$. Therefore $e^{\mathcal{H}}$ is well-defined. The well-definedness of $e^{\mathcal{J}}$ is proved in the same way. \square

The operators $e^{\mathcal{H}}$ and $e^{\mathcal{J}}$ describe the structure of ‘twisted embeddings’. More precisely, we have the following embeddings:

Lemma 5.6. (1) There exists a unique $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module embedding $\mathcal{F}_M: M \otimes \mathcal{F}_{(0,0)}^+ \otimes \mathcal{F}_{(0,0)}^- \rightarrow \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-)$ such that

$$(5.1) \quad v \otimes |(0,0)\rangle^+ \otimes |(0,0)\rangle^- \mapsto e^{\mathcal{H}} \left(\sum_{a \in \mathbb{Z}} v_a \otimes e^{a\alpha^+} \otimes e^{-a\alpha^-} \right)$$

for any $v \in M$.

(2) There exists a unique $V_{\kappa}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module embedding $\mathcal{G}_N: N \otimes \mathcal{F}_{(0,0)}^+ \otimes \mathcal{F}_{(0,0)}^- \rightarrow \iota_{\mathfrak{ns}_2}^*(N \otimes V^+ \otimes V^-)$ such that

$$(5.2) \quad w \otimes |(0,0)\rangle^+ \otimes |(0,0)\rangle^- \mapsto e^{\mathcal{J}} \left(\sum_{a \in \mathbb{Z}} w_a \otimes e^{a\alpha^+} \otimes e^{-a\alpha^-} \right)$$

for any $w \in N$.

Proof. (1) Let $a \in \mathbb{Z}$ and put $e(a) := e^{a\alpha^+} \otimes e^{-a\alpha^-}$. Denote by \tilde{X}_n the linear operator $\iota_{\mathfrak{sl}_2}(\mathbf{X})_{(n)}$ on $\iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-)$, where \mathbf{X} is one of the vectors $\mathbf{H}, \mathbf{E}, \mathbf{F}$ and α^\pm . By computations, we have

$$\begin{aligned} \tilde{H}_n e^{\mathcal{H}}(v_a \otimes e(a)) &= e^{\mathcal{H}}(H_n v_a \otimes e(a)), \\ \tilde{E}_n e^{\mathcal{H}}(v_a \otimes e(a)) &= e^{\mathcal{H}}(E_n v_a \otimes e(a+1)), \\ \tilde{F}_n e^{\mathcal{H}}(v_a \otimes e(a)) &= e^{\mathcal{H}}(F_n v_a \otimes e(a-1)) \end{aligned}$$

for $n \in \mathbb{Z}$. Hence (5.1) defines a $V_k(\mathfrak{sl}_2)$ -module homomorphism

$$(5.3) \quad M \otimes \mathbb{C} |(0,0)\rangle^+ \otimes \mathbb{C} |(0,0)\rangle^- \rightarrow \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-).$$

The injectivity of this mapping follows from the bijectivity of the operator $e^{\mathcal{H}}$ on $M \otimes V^+ \otimes V^-$.

Since we also compute that

$$\begin{aligned}\tilde{b}_n^+ e^{\mathcal{H}}(v_a \otimes e(a)) &= \begin{cases} 0 & \text{if } n > 0 \\ \kappa j e^{\mathcal{H}}(v_a \otimes e(a)) & \text{if } n = 0, \end{cases} \\ \tilde{b}_n^- e^{\mathcal{H}}(v_a \otimes e(a)) &= \begin{cases} 0 & \text{if } n > 0 \\ \kappa j e^{\mathcal{H}}(v_a \otimes e(a)) & \text{if } n = 0 \end{cases}\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$, the mapping (5.3) uniquely extends to the $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module homomorphism $\mathcal{F}_M: M \otimes \mathcal{F}_{(0,0)}^+ \otimes \mathcal{F}_{(0,0)}^- \longrightarrow \iota_{\mathfrak{sl}_2}^*(M \otimes V^+ \otimes V^-)$. Finally, the injectivity of \mathcal{F}_M follows from that of (5.3).

(2) Denote by $\tilde{L}_n, \tilde{J}_n, \tilde{G}_r^\pm$ and \tilde{b}_n^- the linear operators $\iota_{\mathfrak{ns}_2}(\mathbf{L})_{(n)}, \iota_{\mathfrak{ns}_2}(\mathbf{J})_{(n)}, \iota_{\mathfrak{ns}_2}(\mathbf{G}^\pm)_{(r+\frac{1}{2})}$ and $\iota_{\mathfrak{ns}_2}(\boldsymbol{\alpha}^-)_{(n)}$ on $\iota_{\mathfrak{ns}_2}^*(N \otimes V^+ \otimes V^-)$. Similarly to (1), by computations, we have

$$\begin{aligned}\tilde{L}_n e^{\mathcal{J}}(w_a \otimes e(a)) &= e^{\mathcal{J}}(L_n w_a \otimes e(a)), \\ \tilde{J}_n e^{\mathcal{J}}(w_a \otimes e(a)) &= e^{\mathcal{J}}(J_n w_a \otimes e(a)), \\ \tilde{G}_r^+ e^{\mathcal{J}}(w_a \otimes e(a)) &= e^{\mathcal{J}}(G_r^+ w_a \otimes e(a+1)), \\ \tilde{G}_r^- e^{\mathcal{J}}(w_a \otimes e(a)) &= e^{\mathcal{J}}(G_r^- w_a \otimes e(a-1))\end{aligned}$$

for $n \in \mathbb{Z}$. We also have

$$\begin{aligned}\tilde{b}_n^+ e^{\mathcal{J}}(w_a \otimes e(a)) &= \begin{cases} 0 & \text{if } n > 0 \\ \kappa j e^{\mathcal{J}}(w_a \otimes e(a)) & \text{if } n = 0, \end{cases} \\ \tilde{b}_n^- e^{\mathcal{J}}(w_a \otimes e(a)) &= \begin{cases} 0 & \text{if } n > 0 \\ \kappa j e^{\mathcal{J}}(w_a \otimes e(a)) & \text{if } n = 0 \end{cases}\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. Therefore the same discussion as in (1) works. \square

Proposition 5.7. (1) As a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module,

$$\bigoplus_{n,m \in \mathbb{Z}} M^{\theta+n+m} \otimes \mathcal{F}_{(\theta,n)}^+ \otimes \mathcal{F}_{(\theta+n,m)}^- \xrightarrow{\cong} \iota_{\mathfrak{sl}_2}^*(M^\theta \otimes V^+ \otimes V^-).$$

(2) As a $V_{ck}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module,

$$\bigoplus_{n,m \in \mathbb{Z}} N^{\theta+n+m} \otimes \mathcal{F}_{(\theta+m,n)}^+ \otimes \mathcal{F}_{(\theta,m)}^- \xrightarrow{\cong} \iota_{\mathfrak{ns}_2}^*(N^\theta \otimes V^+ \otimes V^-).$$

Proof. (1) Put $h(\theta, n, m) := \frac{\theta+n+m}{2}\mathbf{H} + (\frac{k}{2}\theta - n)\kappa\boldsymbol{\alpha}^+ - (\frac{k}{k+2}n + m)\kappa^{-1}\boldsymbol{\alpha}^-$. Then we have $\iota_{\mathfrak{sl}_2}(h(\theta, n, m)) = \frac{\theta}{2}\mathbf{H} + n\alpha_{-1}^+ + m\alpha_{-1}^-$ and the composition $\iota_{\mathfrak{sl}_2}^*(\text{id}_M \otimes \xi_n^+ \otimes \xi_m^-) \circ \Delta^{h(\theta,n,m)}(\mathcal{F}_M)$ gives an embedding

$$M^{\theta+n+m} \otimes \mathcal{F}_{(\theta,n)}^+ \otimes \mathcal{F}_{(\theta+n,m)}^- \longrightarrow \iota_{\mathfrak{sl}_2}^*(M^\theta \otimes V^+ \otimes V^-),$$

where ξ_n^+ and ξ_m^- are defined in Section B.2. Since each images for $n, m \in \mathbb{Z}$ are in distinct eigenspaces with respect to $\boldsymbol{\alpha}_{(0)}^\pm$, the sum of these embeddings induces an injective homomorphism

$$(5.4) \quad \bigoplus_{n,m \in \mathbb{Z}} M^{\theta+n+m} \otimes \mathcal{F}_{(\theta,n)}^+ \otimes \mathcal{F}_{(\theta+n,m)}^- \longrightarrow \iota_{\mathfrak{sl}_2}^*(M^\theta \otimes V^+ \otimes V^-).$$

By Lemma 5.4, this injection gives an isomorphism.

(2) Put $h'(\theta, n, m) := (\theta + n + m)\mathbf{J} + (\frac{k}{2}(\theta + m) - n)\kappa\alpha^+ - (\frac{k}{k+2}\theta + m)\kappa^{-1}\alpha^-$. Then we have $\iota_{\mathfrak{ns}_2}(h'(\theta, n, m)) = \theta\mathbf{J} + n\alpha_{-1}^+ + m\alpha_{-1}^-$ and the composition $\iota_{\mathfrak{ns}_2}^*(\text{id}_N \otimes \xi_n^+ \otimes \xi_m^-) \circ \Delta^{h'(\theta, n, m)}(\mathcal{G}_N)$ gives rise to an embedding

$$(5.5) \quad \bigoplus_{n, m \in \mathbb{Z}} N^{\theta+n+m} \otimes \mathcal{F}_{(\theta+m, n)}^+ \otimes \mathcal{F}_{(\theta, m)}^- \longrightarrow \iota_{\mathfrak{ns}_2}^*(N^\theta \otimes V^+ \otimes V^-)$$

in the same way as (1). By Lemma 5.4, this gives an isomorphism. \square

5.4. Construction of natural transformation. By the following proposition, we complete the proof of Theorem 4.4.

Proposition 5.8. (1) The assignments

$$\begin{aligned} \mathcal{F}(M) : M &\longrightarrow (\Omega_j^-|) \circ (\Omega_j^+|)(M); v \longmapsto e^{\mathcal{H}} \left(\sum_{a \in \mathbb{Z}} v_a \otimes e^{a\alpha^+} \otimes e^{-a\alpha^-} \right), \\ \mathcal{G}(N) : N &\longrightarrow (\Omega_j^+|) \circ (\Omega_j^-|)(N); w \longmapsto e^{\mathcal{J}} \left(\sum_{a \in \mathbb{Z}} w_a \otimes e^{a\alpha^+} \otimes e^{-a\alpha^-} \right) \end{aligned}$$

give a $V_k(\mathfrak{sl}_2)$ -module isomorphism and a $V_{c_k}(\mathfrak{ns}_2)$ -module isomorphism, respectively.

(2) The sets of isomorphisms \mathcal{F} and \mathcal{G} give natural transformations $\text{Id}_{\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h, j)}} \cong (\Omega_j^-|) \circ (\Omega_j^+|)$ and $\text{Id}_{\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{(h - \frac{j^2}{k+2}, \frac{2j}{k+2})}} \cong (\Omega_j^+|) \circ (\Omega_j^-|)$, respectively.

Proof. (1) Both statements follow from Lemma 5.1, 5.6 and Proposition 5.7.

(2) Let $f \in \text{Hom}_{V_k(\mathfrak{sl}_2)}(M, M')$ and $g \in \text{Hom}_{V_{c_k}(\mathfrak{ns}_2)}(N, N')$. It suffices to show that the following diagrams

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \mathcal{F}(M) \downarrow & & \downarrow \mathcal{F}(M') \\ \Omega_j^{\mathfrak{sl}_2}(M) & \xrightarrow{\Omega_j^{\mathfrak{sl}_2}(f)} & \Omega_j^{\mathfrak{sl}_2}(M') \end{array} \quad \begin{array}{ccc} N & \xrightarrow{g} & N' \\ \mathcal{G}(N) \downarrow & & \downarrow \mathcal{G}(N') \\ \Omega_j^{\mathfrak{ns}_2}(N) & \xrightarrow{\Omega_j^{\mathfrak{ns}_2}(g)} & \Omega_j^{\mathfrak{ns}_2}(N') \end{array}$$

commute. Since f commutes with the action of $V_k(\mathfrak{sl}_2)$, we have $(f(v))_a = f(v_a)$ for $v \in M$ and $a \in \mathbb{Z}$. It also follows that the operators $\Omega_j^{\mathfrak{sl}_2}(f)$ and $e^{\mathcal{H}}$ commute in $\text{End}(M \otimes V^+ \otimes V^-)$. Then the former diagram commutes. The latter is proved in the same way. \square

6. SPECTRAL FLOW EQUIVARIANCE

In this section we will discuss the relationship between the functors Ω_j^\pm and the spectral flow automorphisms. To simplify notation, we put

$$\begin{aligned} h(\theta, r) &:= (\theta + r)\mathbf{J} + \left(\frac{k}{2}\theta - r \right) \kappa\alpha^+ \in V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+, \\ h'(\theta, r) &:= \frac{\theta + r}{2}\mathbf{H} - \left(\frac{k}{k+2}\theta + r \right) \kappa^{-1}\alpha^- \in V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^- \end{aligned}$$

for $\theta, r \in \mathbb{Z}$.

Proposition 6.1. (1) As a $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module,

$$(6.1) \quad \bigoplus_{n \in \mathbb{Z}} \Omega_j^+(M)^n \otimes \mathcal{F}_{(0,n)}^+ \cong \iota_+^*(M \otimes V^+).$$

(2) As a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^-$ -module,

$$(6.2) \quad \bigoplus_{m \in \mathbb{Z}} \Omega_j^-(N)^m \otimes \mathcal{F}_{(0,m)}^- \cong \iota_-^*(N \otimes V^-).$$

Proof. (1) By using Lemma 4.3 (1), it suffices to show that $\Omega_j^+(M)^n \cong \Omega_{j-n}^+(M)$ as a $V_{c_k}(\mathfrak{ns}_2)$ -module. Let $r \in \mathbb{Z}$ and denote by f_M the isomorphism in (4.2). Since $\iota_+(h(0, r)) = r\alpha_{-1}^+ \in V_k(\mathfrak{sl}_2) \otimes V^+$, we get the $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module isomorphism

$$\Delta^{h(0,r)}(f_M): \bigoplus_{n \in \mathbb{Z}} \Omega_{j-n}^+(M)^r \otimes \mathcal{F}_{(0,n+r)}^+ \xrightarrow{\cong} \iota_+^*(M \otimes (V^+)^r).$$

By using the V^+ -module isomorphism $\xi_r^+: (V^+)^r \xrightarrow{\cong} V^+$, the composition $(f_M)^{-1} \circ \iota_+^*(\text{id}_M \otimes \xi_r^+) \circ \Delta^{h(0,r)}(f_M)$ gives a $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module isomorphism

$$\bigoplus_{n \in \mathbb{Z}} \Omega_{j-n}^+(M)^r \otimes \mathcal{F}_{(0,n+r)}^+ \xrightarrow{\cong} \bigoplus_{n \in \mathbb{Z}} \Omega_{j-n-r}^+(M) \otimes \mathcal{F}_{(0,n+r)}^+.$$

The restriction of this mapping to the invariant subspace $\Omega_j^+(M)^r \otimes \mathbb{C}|(0, r)\rangle^+$ gives a $V_{c_k}(\mathfrak{ns}_2)$ -module isomorphism

$$\Omega_j^+(M)^r \otimes \mathbb{C}|(0, r)\rangle^+ \xrightarrow{\cong} \Omega_{j-r}^+(M) \otimes \mathbb{C}|(0, r)\rangle^+.$$

This completes the proof.

(2) Applying the functor $\Delta^{h'(0,r)}$ to the isomorphism (4.3), we also get the isomorphism

$$\bigoplus_{m \in \mathbb{Z}} \Omega_{j+\frac{k+2}{2}m}^-(N)^r \otimes \mathcal{F}_{(0,m+r)}^- \xrightarrow{\cong} \iota_-^*(N \otimes (V^-)^r).$$

Since $(V^-)^r \cong V^-$ as a V^- -module, we obtain

$$\Omega_j^-(N)^r \otimes \mathbb{C}|(0, r)\rangle^- \xrightarrow{\cong} \Omega_{j+\frac{k+2}{2}r}^-(N) \otimes \mathbb{C}|(0, r)\rangle^-$$

in the same way as (1). \square

Corollary 6.2. (1) As a $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module,

$$\bigoplus_{n \in \mathbb{Z}} \Omega_j^+(M)^{\theta+n} \otimes \mathcal{F}_{(\theta,n)}^+ \cong \iota_+^*(M^\theta \otimes V^+).$$

(2) As a $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^-$ -module,

$$\bigoplus_{m \in \mathbb{Z}} \Omega_j^-(N)^{\theta+m} \otimes \mathcal{F}_{(\theta,m)}^- \cong \iota_-^*(N^\theta \otimes V^-).$$

Proof. The statement follows by applying the functors $\Delta^{h(\theta,0)}$ and $\Delta^{h'(\theta,0)}$ to the isomorphisms (6.1) and (6.2), respectively. \square

Then we obtain the following “spectral flow equivariance” property.

Corollary 6.3. For $a, b \in \mathbb{Z}$, the following diagrams

$$\begin{array}{ccc}
\mathcal{LW}_{V_{\mathfrak{sl}_2}}^{\overline{(h,j;0)}} & \xrightarrow{\Omega_j^+} & \mathcal{LW}_{V_{\mathfrak{ns}_2}}^{\overline{(h,j;0)}} \\
\Delta_{\mathfrak{sl}_2}^a \downarrow & & \downarrow \Delta_{\mathfrak{ns}_2}^{a+b} \\
\mathcal{LW}_{V_{\mathfrak{sl}_2}}^{\overline{(h,j;a)}} & \xrightarrow{\Omega_j^+ + \frac{k}{2}a-b} & \mathcal{LW}_{V_{\mathfrak{ns}_2}}^{\overline{(h,j;a+b)}} \\
\\
\mathcal{LW}_{V_{\mathfrak{ns}_2}}^{\overline{(h,j;0)}} & \xrightarrow{\Omega_j^-} & \mathcal{LW}_{V_{\mathfrak{sl}_2}}^{\overline{(h,j;0)}} \\
\Delta_{\mathfrak{ns}_2}^a \downarrow & & \downarrow \Delta_{\mathfrak{sl}_2}^{a+b} \\
\mathcal{LW}_{V_{\mathfrak{ns}_2}}^{\overline{(h,j;a)}} & \xrightarrow{\Omega_j^- + \frac{k}{2}a + \frac{k+2}{2}b} & \mathcal{LW}_{V_{\mathfrak{sl}_2}}^{\overline{(h,j;a+b)}}
\end{array}$$

commute up to natural equivalence.

Example 6.4. Setting $a = 0$ in the former diagram, we obtain $\Omega_{j-b}^+ \cong \Delta_{\mathfrak{ns}_2}^b \circ \Omega_j^+$ as a functor from $\mathcal{LW}_{V_{\mathfrak{sl}_2}}^{\overline{(h,j)}}$ to $\mathcal{LW}_{V_{\mathfrak{ns}_2}}$.

7. APPLICATIONS

7.1. Examples of corresponding pairs. In this subsection we show that relaxed highest weight $\widehat{\mathfrak{sl}_2}$ -modules of level k exactly correspond to highest weight \mathfrak{ns}_2 -modules of central charge c_k . To simplify notation, we write $\mathcal{M}_{(h,j,k)}$ and $\mathcal{L}_{(h,j,k)}$ for $\mathcal{M}_{h-\frac{j^2}{k+2}, \frac{2j}{k+2}, c_k}$ and $\mathcal{L}_{h-\frac{j^2}{k+2}, \frac{2j}{k+2}, c_k}$, respectively.

Proposition 7.1. We have the following isomorphisms:

- (1) $\Omega_j^+(R_{h,j,k}) \cong \mathcal{M}_{(h,j,k)}$,
- (2) $\Omega_j^+(M_{j,k}) \cong \mathcal{M}_{\frac{2j}{k+2}, c_k}^+$,
- (3) $\Omega_0^+(V_k(\mathfrak{sl}_2)) \cong V_{c_k}(\mathfrak{ns}_2)$,
- (4) $\Omega_j^+(L_{h,j,k}) \cong \mathcal{L}_{(h,j,k)}$.

Proof. Let $\theta \in \mathbb{Z}$. Denote by $|h, j, k; \theta\rangle$ and $|(h, j, k); \theta\rangle^{\mathfrak{ns}_2}$ the canonical generators of $R_{h,j,k}^\theta$ and $\mathcal{M}_{(h,j,k)}^\theta$, respectively. By the annihilation relations, the $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+$ -module homomorphism

$$f(\theta): \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_{(h,j,k)}^{\theta+n} \otimes \mathcal{F}_{(\theta,n)}^+ \longrightarrow \iota_+^*(R_{h,j,k}^\theta \otimes V^+)$$

is uniquely determined by sending

$$|(h, j, k); \theta + n\rangle^{\mathfrak{ns}_2} \otimes |(\theta, n)\rangle^+ \longmapsto |h, j, k; \theta\rangle \otimes e^{n\alpha^+}$$

for $n \in \mathbb{Z}$. In the same way, the $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^-$ -module homomorphism

$$g(\theta): \bigoplus_{m \in \mathbb{Z}} R_{h,j,k}^{\theta+m} \otimes \mathcal{F}_{(\theta,m)}^- \longrightarrow \iota_-^*(\mathcal{M}_{(h,j,k)}^\theta \otimes V^-).$$

is uniquely determined by sending

$$|h, j, k; \theta + m\rangle \otimes |(\theta, m)\rangle^- \longmapsto |(h, j, k); \theta\rangle^{\mathfrak{ns}_2} \otimes e^{m\alpha^-}$$

for $m \in \mathbb{Z}$. Since we have

$$e^{\mathcal{H}}(|h, j, k; \theta\rangle \otimes e^{n\alpha^+} \otimes e^{m\alpha^-}) = |h, j, k; \theta\rangle \otimes e^{n\alpha^+} \otimes e^{m\alpha^-}$$

for any $n, m \in \mathbb{Z}$, the $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module homomorphism

$$\begin{aligned} & \bigoplus_{n, m \in \mathbb{Z}} R_{h, j, k}^{\theta+n+m} \otimes \mathcal{F}_{(\theta, n)}^+ \otimes \mathcal{F}_{(\theta+n, m)}^- \\ & \xrightarrow{\oplus g(\theta+n)} (\iota_-)^*_{13} \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{M}_{(h, j, k)}^{\theta+n} \otimes \mathcal{F}_{(\theta, n)}^+ \otimes V^- \right) \\ & \xrightarrow{f(\theta)} (\iota_-)^*_{13} \circ (\iota_+)^*_{12} (R_{h, j, k}^\theta \otimes V^+ \otimes V^-) \end{aligned}$$

coincides with the $V_k(\mathfrak{sl}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module isomorphism (5.4). Therefore $g(\theta)$ is injective and $f(\theta)$ is surjective for any $\theta \in \mathbb{Z}$. Similarly, since

$$e^{\mathcal{J}}(|(h, j, k); \theta\rangle^{\text{ns}} \otimes e^{n\alpha^+} \otimes e^{m\alpha^-}) = |(h, j, k); \theta\rangle^{\text{ns}} \otimes e^{n\alpha^+} \otimes e^{m\alpha^-},$$

the $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module homomorphism

$$\begin{aligned} & \bigoplus_{n, m \in \mathbb{Z}} \mathcal{M}_{(h, j, k)}^{\theta+n+m} \otimes \mathcal{F}_{(\theta+m, n)}^+ \otimes \mathcal{F}_{(\theta, m)}^- \\ & \xrightarrow{\oplus f(\theta+m)} (\iota_+)^*_{12} \left(\bigoplus_{m \in \mathbb{Z}} R_{(h, j, k)}^{\theta+m} \otimes V^+ \otimes \mathcal{F}_{(\theta, m)}^- \right) \\ & \xrightarrow{g(\theta)} (\iota_+)^*_{12} \circ (\iota_-)^*_{13} (\mathcal{M}_{(h, j, k)}^\theta \otimes V^+ \otimes V^-) \end{aligned}$$

coincides with the $V_{c_k}(\mathfrak{ns}_2) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-$ -module isomorphism (5.5). Therefore $g(\theta)$ is surjective and $f(\theta)$ is injective. This completes the proof of (1). Since (2) and (3) are proved in the same way, we omit the proof.

Finally, as the functor Ω_j^\pm gives a categorical equivalence, the module $\Omega_j^+(L_{h, j, k})$ is isomorphic to the simple quotient of $\Omega_j^+(R_{h, j, k}) \cong \mathcal{M}_{(h, j, k)}$. \square

Remark 7.2. It is known that the VOA $V_k(\mathfrak{sl}_2)$ has a non-trivial ideal if and only if k is an admissible level for $\widehat{\mathfrak{sl}}_2$, i.e. there exists a pair of coprime integers $(p, p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ such that $k = -2 + \frac{p}{p'}$ (see [GK07, Theorem 0.2.1]). It follows from Proposition 7.1 that the VOSA $V_c(\mathfrak{ns}_2)$ has a non-trivial ideal if and only if there exists an admissible level k for $\widehat{\mathfrak{sl}}_2$ such that $c = c_k = 3(1 - \frac{2p'}{p})$. This is already proved by M. Gorelik and V. Kac [GK07, Corollary 9.1.5 (ii)] and we give another proof.

Let the pair $(V_{\mathfrak{sl}_2}, V_{\mathfrak{ns}_2})$ stands for $(V_k(\mathfrak{sl}_2), V_{c_k}(\mathfrak{ns}_2))$ or $(L_k(\mathfrak{sl}_2), L_{c_k}(\mathfrak{ns}_2))$. By definition of the functors Ω_0^\pm , we obtain the following:

Corollary 7.3. The coset vertex superalgebras $C(V_{\mathfrak{sl}_2} \otimes V^+, \iota_+(\mathcal{F}^+))$ and $C(V_{\mathfrak{ns}_2} \otimes V^-, \iota_-(\mathcal{F}^-))$ are isomorphic to $V_{\mathfrak{ns}_2}$ and $V_{\mathfrak{sl}_2}$, respectively.

Remark 7.4. The former isomorphism $C(L_k(\mathfrak{sl}_2) \otimes V^+, \iota_+(\mathcal{F}^+)) \cong L_{c_k}(\mathfrak{ns}_2)$ is well known. For example, see [CL14, Lemma 8.7 and 8.8].

Corollary 7.5. Assume that $k \neq 0$. Then $\{\mathcal{L}_{h, j, c_k} \mid h, j \in \mathbb{C}\}$ provides the complete representatives of the equivalence classes of simple objects in $\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}$.

Proof. Let L be a simple object in $\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}$. By Fact 3.6, Corollary 6.3 and Proposition 7.1 (4), there exist $h', j' \in \mathbb{C}$ and $\theta \in \mathbb{Z}$ such that $L \cong \mathcal{L}_{h', j', c_k}^\theta$. Since the set of eigenvalues with respect to L_0 on $\mathcal{L}_{h', j', c_k}^\theta$ is lower bounded, there exist unique $h, j \in \mathbb{C}$ such that $\mathcal{L}_{h', j', c_k}^\theta \cong \mathcal{L}_{h, j, c_k}$. \square

7.2. Application 1: Equivalence in the simple case. As a corollary of Proposition 7.1 (4), we get another proof of the following theorem obtained by [Ada99, Theorem 4.1 and 5.1].

Theorem 7.6. The homomorphisms ι_+ and ι_- in Theorem 2.2 factor through the corresponding simple quotients.

Proof. By Proposition 7.1 (4), $\Omega_0^+(L_k(\mathfrak{sl}_2))$ and $\Omega_0^-(L_{c_k}(\mathfrak{ns}_2))$ are isomorphic to $L_{c_k}(\mathfrak{ns}_2)$ and $L_k(\mathfrak{sl}_2)$, respectively. Then the restrictions of the isomorphisms (4.2) and (4.3) to the 0-th components is identified with the above mappings ι_+ and ι_- , respectively. \square

In a similar way, we get the simple quotient version of Theorem 4.4.

Theorem 7.7. Assume that $k \neq 0$. The restrictions of $\Omega_j^+|$ and $\Omega_j^-|$ in Theorem 4.4 also give the categorical equivalences $\mathcal{LW}_{L_k(\mathfrak{sl}_2)}^{(h,j)} \xrightarrow{\cong} \mathcal{LW}_{L_{c_k}(\mathfrak{ns}_2)}^{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})}$ and $\mathcal{LW}_{L_{c_k}(\mathfrak{ns}_2)}^{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})} \xrightarrow{\cong} \mathcal{LW}_{L_k(\mathfrak{sl}_2)}^{(h,j)}$, respectively.

7.3. Application 2: Character formulae.

7.3.1. At general levels. In this subsection we fix $h, j \in \mathbb{C}$ and suppose that

$$M \in \text{Obj} \left(\mathcal{LW}_{V_k(\mathfrak{sl}_2)}^{(h,j)} \right) \quad \text{and} \quad N \in \text{Obj} \left(\mathcal{LW}_{V_{c_k}(\mathfrak{ns}_2)}^{(h-\frac{j^2}{k+2}, \frac{2j}{k+2})} \right).$$

Denote by $M_n(q)$ and $N_n(q)$ the elements of $\mathbb{C}((q))$ such that

$$\text{ch}(M)(q, x) = q^h x^j \sum_{n \in \mathbb{Z}} M_n(q) x^n, \quad \text{ch}(N)(q, x) = q^{h-\frac{j^2}{k+2}} x^{\frac{2j}{k+2}} \sum_{n \in \mathbb{Z}} N_n(q) x^n.$$

Theorem 7.8. We have the following:

- (1) $\text{ch}(\Omega_j^+(M))(q, x) = q^{h-\frac{j^2}{k+2}} x^{\frac{2j}{k+2}} \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} M_n(q) x^n.$
- (2) $\text{ch}(\Omega_j^-(N))(q, x) = q^h x^j \sum_{n \in \mathbb{Z}} q^{-\frac{n^2}{2}} N_n(q) x^n.$

Proof. (1) By Proposition 6.1.1 and easy calculation, we have

$$\begin{aligned} (7.1) \quad \text{ch}(\iota_+^*(M \otimes V^+))(q, x_1, x_2) &= \sum_{n \in \mathbb{Z}} \text{ch}(\Omega_j^+(M)^n)(q, x_1) \text{ch}(\mathcal{F}_{\kappa(j-n)}^+)(q, x_2) \\ &= \text{ch}(M)(q, x_1^{\kappa^2} x_2^{\kappa}) \text{ch}(V^+)(q, x_1^{1-\kappa^2} x_2^{-\kappa}). \end{aligned}$$

By the character formula for fermionic Fock module, we have

$$\text{ch}(V^+)(q, x_1^{1-\kappa^2} x_2^{-\kappa}) = \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2}} x_1^{(1-\kappa^2)m} x_2^{-\kappa m} \times (q; q)_{\infty}^{-1}.$$

Multiplying $q^{-\frac{j^2}{k+2}}(q; q)_{\infty}$ to (7.1) and comparing the coefficients of x_2^j , we obtain the required formula.

(2) The equality follows from (1) and $\Omega_j^+ \circ \Omega_j^-(N) \cong N.$ \square

Remark 7.9. The character formula for irreducible highest weight $\widehat{\mathfrak{sl}}_2$ -modules at non-critical level (*i.e.* $k \neq -2$) is given by [KT00, Theorem 1.1]. Therefore we obtain all the corresponding character formula for the irreducible quotients of chiral Verma modules.

7.3.2. At admissible levels. We fix a pair of coprime integers $(p, p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ and a pair of integers (r, s) such that $1 \leq r \leq p-1$ and $0 \leq s \leq p'-1$. Throughout this subsection, we always assume that $k = -2 + \frac{p}{p'}$. It follows that $c_k = 3 \left(1 - \frac{2p'}{p}\right)$.

We use the following notations:

$$M(n, m) := M_{\frac{n-1}{2} - \frac{p}{p'} \frac{m}{2}, k}, \quad \mathcal{M}^+(n, m) := \mathcal{M}_{\frac{p'}{p}(n-1) - m, c_k}^+$$

for $n, m \in \mathbb{Z}$. Denote by $L(n, m)$ and $\mathcal{L}(n, m)$ the simple quotients of $M(n, m)$ and $\mathcal{M}^+(n, m)$, respectively. Then the following resolution is obtained by F. Malikov through the analysis of singular vectors in Verma modules over $\widehat{\mathfrak{sl}}_2$.

Fact 7.10. [Mal91, Theorem A] We set $M_n := M(n) \oplus M(-n)$ for $n > 0$, where $M(2m) := M(2pm + r, s)$ and $M(2m-1) := M(2pm - r, s)$ for $m \in \mathbb{Z}$. Then there exists an exact sequence

$$0 \leftarrow L(r, s) \leftarrow M(r, s) \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$$

Applying the functor $\Omega_{\frac{r-1}{2} - \frac{p}{p'} \frac{s}{2}}^+$ to the above resolution, we obtain by Corollary 6.3 another proof of the following BGG-type resolution for the $\mathcal{N} = 2$ SCA module $\mathcal{L}(r, s)$ given by [FSST99, Theorem 3.1].

Theorem 7.11. We set $\mathcal{M}_n^+ := \mathcal{M}^+(n) \oplus \mathcal{M}^+(-n)$ for $n > 0$, where $\mathcal{M}^+(2m) := \mathcal{M}^+(2pm + r, s)^{pm}$ and $\mathcal{M}^+(2m-1) := \mathcal{M}^+(2pm - r, s)^{pm-r}$ for $m \in \mathbb{Z}$. Then there exists an exact sequence

$$0 \leftarrow \mathcal{L}(r, s) \leftarrow \mathcal{M}^+(r, s) \leftarrow \mathcal{M}_1^+ \leftarrow \mathcal{M}_2^+ \leftarrow \cdots \leftarrow \mathcal{M}_n^+ \leftarrow \cdots$$

Remark 7.12. (1) The proof in [FSST99] is essentially based on the ‘equivalence’ in [FST98, Theorem IV.3]. However, an explicit construction of the categorical equivalence is not given in the literature to the best of our knowledge.

(2) When $p' = 1$, every irreducible unitarizable highest weight \mathfrak{ns}_2 -module of central charge $c = 3(1 - \frac{2}{p})$ is of the form $\mathcal{L}(r, 0)^\theta$ for some $1 \leq r \leq p-1$ and $0 \leq \theta \leq r-1$. The highest weight (h, j, c) of $\mathcal{L}(r, 0)^\theta$ is given by

$$(h, j) = \left(\frac{(r - \theta - \frac{1}{2})(\theta + \frac{1}{2}) - \frac{1}{4}}{p}, \frac{(r - \theta - \frac{1}{2}) - (\theta + \frac{1}{2})}{p} \right).$$

We set $\vartheta(q, z) := (-zq^{\frac{1}{2}}; q)_\infty (-z^{-1}q^{\frac{1}{2}}; q)_\infty (q; q)_\infty$ and $\eta(q) := q^{\frac{1}{24}}(q; q)_\infty$. Then, as a corollary of Proposition 7.11, we reprove the following character formula for $\widetilde{\text{ch}}(\mathcal{L}(r, s)) := q^{-\frac{c_k}{24}} \text{ch}(\mathcal{L}(r, s))$ obtained by [FSST99, Theorem 4.8].

Theorem 7.13. Put $j = \frac{r-1}{2} - \frac{p}{p'} \frac{s}{2}$. Then we have

$$\widetilde{\text{ch}}(\mathcal{L}(r, s)) = q^{-\frac{p'}{p}j^2} x^{\frac{2p'}{p}j} \frac{\vartheta(q, x)}{\eta(q)^3} \Phi_{p, p'; r, s}(q, x),$$

where $\Phi_{p, p'; r, s}(q, x)$ is the expansion of the meromorphic function

$$\sum_{n \in \mathbb{Z}} \left(\frac{q^{\frac{p'}{p}(pn + \frac{1}{2} + j)^2}}{1 + xq^{pn + \frac{1}{2}}} - \frac{q^{\frac{p'}{p}(pn - r + \frac{1}{2} + j)^2}}{1 + xq^{pn - r + \frac{1}{2}}} \right)$$

in the region \mathbb{A} (see Remark 3.7 for the definition).

Proof. By the Poincaré-Birkoff-Witt theorem and Lemma 5.2, we have

$$q^{-\frac{c_k}{24}} \text{ch}(\mathcal{M}^+(2\theta + r, s)^\theta) = q^{-\frac{p'}{p}j^2} x^{\frac{2p'}{p}j} \frac{\vartheta(q, x)}{\eta(q)^3} q^{\frac{p'}{p}(\theta + \frac{1}{2} + j)^2} \frac{1}{1 + xq^{\theta + \frac{1}{2}}}$$

for any $\theta \in \mathbb{Z}$. Thus the formula follows from the BGG-type resolution. \square

Remark 7.14. The formal series

$$\Phi_{p,p';r,s}(q, x) = \left(\sum_{n,m \geq 0} - \sum_{n,m < 0} \right) (-x)^m \varphi_{p,p';r,s}^{n,m}(q),$$

where

$$\varphi_{p,p';r,s}^{n,m}(q) := q^{\frac{p'}{p}(pn + \frac{1}{2} + j)^2 + (pn + \frac{1}{2})m} - q^{\frac{p'}{p}(pn + p - r + \frac{1}{2} + j)^2 + (pn + p - r + \frac{1}{2})m},$$

is absolutely convergent in the region \mathbb{A} . Note that this function is related with a certain specialization of the mock theta function of type $A(1, 0)$ defined in [KW14]. See [KW14, Proposition 9.2] for example.

APPENDIX A. VERTEX SUPERALGEBRAS

In this section we recall some facts about conformal vertex superalgebras and their modules to fix our notation. See [Kac98] for the details.

A.1. Definitions. Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a \mathbb{Z}_2 -graded \mathbb{C} -vector space, $Y(-; z): V \rightarrow \text{End}_{\mathbb{C}}(V)[[z^{\pm 1}]]$ an even linear mapping and $\mathbf{1} \in V^{\bar{0}}$ a non-zero vector. In what follows, we write

$$Y(A; z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1} \in \text{End}_{\mathbb{C}}(V)[[z^{\pm 1}]]$$

for $A \in V$.

Definition A.1. A triple $(V, Y, \mathbf{1})$ is a *vertex superalgebra* if

- (1) $A_{(n)}B = 0$ for any $A, B \in V$ and $n \gg 0$,
- (2) $\mathbf{1}_{(n)}A = \delta_{n,-1}A$ for any $A \in V$ and $n \in \mathbb{Z}$,
- (3) $A_{(-1)}\mathbf{1} = A$ for any $A \in V$,
- (4) the *Borcherds identity* holds, i.e.

$$\begin{aligned} & \sum_{i \geq 0} \binom{p}{i} (A_{(r+i)}B)_{(p+q-i)} \\ &= \sum_{i \geq 0} (-1)^i \binom{r}{i} (A_{(p+r-i)}B_{(q+i)} - (-1)^{r+ab} B_{(q+r-i)}A_{(p+i)}) \end{aligned}$$

for any $A \in V^{\bar{a}}$, $B \in V^{\bar{b}}$ and $p, q, r \in \mathbb{Z}$.

Moreover, a vertex superalgebra $(V, Y, \mathbf{1})$ is \mathbb{Q} -graded if

- (1) V is a \mathbb{Q} -graded \mathbb{C} -vector space, i.e. $V = \bigoplus_{\Delta \in \mathbb{Q}} V_{\Delta}$,
- (2) $(V_{\Delta})_{(n)}(V_{\Delta'}) \subset V_{\Delta + \Delta' - n - 1}$ for any $\Delta, \Delta' \in \mathbb{Q}$ and $n \in \mathbb{Z}$.

A non-zero even vector $\omega \in V_2$ is a *conformal vector of central charge $c \in \mathbb{C}$* if

- (1) The Virasoro algebra $Vir = \bigoplus_{n \in \mathbb{Z}} L_n \oplus \mathbb{C}C$ acts on V , where the action is given by $L_n \mapsto \omega_{(n+1)}$ and $C \mapsto c \text{id}_V$.
- (2) For any $A \in V$, we have $Y(\omega_{(0)}A; z) = \partial_z Y(A; z)$.
- (3) The operator $\omega_{(1)}$ is diagonalizable and the corresponding eigenspace decomposition coincides with the \mathbb{Q} -grading of V .

In this paper, a \mathbb{Q} -graded vertex superalgebra together with a conformal vector of central charge $c \in \mathbb{C}$ is called a *conformal vertex superalgebra* of central charge c . By abuse of notation, we write L_n for the linear operator $\omega_{(n+1)} \in \text{End}(V)$.

Remark A.2. A conformal vertex superalgebra $(V, Y, \mathbf{1}, \omega)$ is called a *vertex operator superalgebra* (VOSA) if $\dim V_\Delta < \infty$ for any $\Delta \in \mathbb{Q}$ and $V_\Delta = \{0\}$ for $\Delta \notin \frac{1}{2}\mathbb{Z}$ or $\Delta \ll 0$. A VOSA $(V, Y, \mathbf{1}, \omega)$ is simply called a *vertex operator algebra* (VOA) if $V = V^{\bar{0}}$ and $V_\Delta = \{0\}$ for $\Delta \notin \mathbb{Z}$.

Definition A.3. Let $(V^1, Y^1, \mathbf{1}^1, \omega^1)$ and $(V^2, Y^2, \mathbf{1}^2, \omega^2)$ be conformal vertex superalgebras. A linear mapping $f: V^1 \rightarrow V^2$ is a *morphism of vertex superalgebra* if $f(\mathbf{1}^1) = \mathbf{1}^2$, $f(\omega^1) = \omega^2$ and $f(Y^1(A; z)B) = Y^2(f(A; z)f(B))$ for any $A, B \in V^1$.

Definition A.4. Let V be a vertex superalgebra and W its vertex subsuperalgebra. The *commutant* (or *coset*) *vertex superalgebra* $C(V, W)$ is the vertex subsuperalgebra of V defined by

$$C(V, W) := \{v \in V \mid v_{(n)}w = 0 \text{ for any } w \in W \text{ and } n \geq 0\}$$

together with $Y(-; z)|_{C(V, W)}$ and $\mathbf{1} \in C(V, W)$.

A.2. Weak modules. Let $M = M^{\bar{0}} \oplus M^{\bar{1}}$ be a \mathbb{Z}_2 -graded \mathbb{C} -vector space and $Y_M(-; z): V \rightarrow \text{End}_{\mathbb{C}}(M)[[z^{\pm 1}]]$ an even linear mapping. Similarly to the case of the state-field correspondence $Y(-; z)$, we write

$$Y_M(A; z) = \sum_{n \in \mathbb{Z}} A_{(n)}^M z^{-n-1} \in \text{End}_{\mathbb{C}}(M)[[z^{\pm 1}]]$$

for $A \in V$.

Definition A.5. A pair (M, Y_M) is a *weak V -module* if

- (1) $A_{(n)}^M m = 0$ for any $A \in V$, $m \in M$ and $n \gg 0$,
- (2) $\mathbf{1}_{(n)}^M m = \delta_{n, -1} m$ for any $m \in M$ and $n \in \mathbb{Z}$,
- (3) the Borcherds identity holds, *i.e.*

$$\begin{aligned} & \sum_{i \geq 0} \binom{p}{i} (A_{(r+i)} B)_{(p+q-i)}^M \\ &= \sum_{i \geq 0} (-1)^i \binom{r}{i} \left(A_{(p+r-i)}^M B_{(q+i)}^M - (-1)^{r+ab} B_{(q+r-i)}^M A_{(p+i)}^M \right) \end{aligned}$$

for any $A \in V^{\bar{a}}$, $B \in V^{\bar{b}}$ and $p, q, r \in \mathbb{Z}$.

Definition A.6. Let (M^1, Y_{M^1}) and (M^2, Y_{M^2}) be weak V -modules. A linear mapping $f: M^1 \rightarrow M^2$ is a *morphism of weak V -modules* if $f(Y_{M^1}(A; z)m) = Y_{M^2}(A; z)f(m)$ for any $A \in V$ and $m \in M^1$.

We denote the category of weak V -modules by $V\text{-Mod}$. By definition, the pair (V, Y) is a weak V -module called the *adjoint module*.

APPENDIX B. SPECTRAL FLOW AUTOMORPHISMS

In this section we identify spectral flow twisted modules over $\widehat{\mathfrak{sl}}_2$ and \mathfrak{ns}_2 in [FST98] with Li's Δ -twisted modules in [Li97]. Moreover, we consider analogues of the spectral flows in the Heisenberg VOAs \mathcal{F}^\pm and lattice conformal vertex superalgebras V^\pm .

B.1. Li's Δ -automorphisms. In this subsection, V always stands for $V_k(\mathfrak{sl}_2)$, $L_k(\mathfrak{sl}_2)$, $V_{c_k}(\mathfrak{ns}_2)$, $L_{c_k}(\mathfrak{ns}_2)$, \mathcal{F}^\pm or V^\pm . At first, we recall Li's theory in the super case. According to [Li97], we consider $\Delta(z) \in \text{End}(V)\{z\}$ which satisfies the following conditions:

$$\begin{aligned} \Delta(z)v &\in V[z, z^{-1}] \text{ for any } v \in V, \\ \Delta(z)\mathbf{1} &= \mathbf{1}, \\ [L_{-1}, \Delta(z)] &= -\partial_z \Delta(z), \\ Y(\Delta(z+w)v; z)\Delta(w) &= \Delta(w)Y(v; z). \end{aligned}$$

Let us denote by $G(V)$ the set of elements of $\text{End}(V)\{z\}$ satisfying the above conditions.

For simplicity, we put $h^{V_{\mathfrak{sl}_2}} := \frac{\theta}{2}\mathbf{H}$, $h^{V_{\mathfrak{ns}_2}} := \theta\mathbf{J}$, $h^{\mathcal{F}^\pm} := \lambda\alpha^\pm$ and $h^{V^\pm} := r\alpha_{-1}^\pm$ for $\theta, r \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$.

Definition B.1. We define

$$\Delta(h^V; z) := z^{h_{(0)}^V} \exp\left(\sum_{j=1}^{\infty} \frac{h_{(j)}^V}{-j} (-z)^{-j}\right) \in \text{End}(V)[[z, z^{-1}]].$$

Lemma B.2. For any $\theta, r \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, we have $\Delta(h^V; z) \in G(V)$.

Proof. Note that $h_{(0)}^V$ has integral eigenvalues on V . Though the vertex superalgebras are not regular, the same discussion in the proof of [Li97, Proposition 3.2] still works. (See also [Ada01, Proposition 2.1].) \square

For a weight-wise admissible (V, S_V) -module (M, Y_M) , the mapping

$$Y_M(\Delta(h^V; z)-; z) : V \rightarrow \text{End}(M)[[z, z^{-1}]]$$

also defines another weight-wise admissible (V, S_V) -module structure on M .

Definition B.3. The assignments

$$\begin{aligned} \text{Obj}(\mathcal{LW}_V) &\rightarrow \text{Obj}(\mathcal{LW}_V); M \mapsto \Delta^h(M) := (M, Y_M(\Delta(h; z)-; z)), \\ \text{Hom}_V(M, N) &\rightarrow \text{Hom}_V(\Delta^h(M), \Delta^h(N)); f \mapsto f \end{aligned}$$

define a functor $\Delta^h : \mathcal{LW}_V \rightarrow \mathcal{LW}_V$.

To simplify notation, we put $\Delta_{\mathfrak{sl}_2}^\theta := \Delta^{\frac{\theta}{2}\mathbf{H}}$, $M^\theta := \Delta^{\frac{\theta}{2}\mathbf{H}}(M)$, $\Delta_{\mathfrak{ns}_2}^\theta := \Delta^{\theta\mathbf{J}}$ and $N^\theta := \Delta^{\theta\mathbf{J}}(N)$.

B.2. Heisenberg and lattice case. By some computations, it follows that there exist a unique \mathcal{F}^\pm -module isomorphisms such that $\Delta^{\pm\lambda\alpha^\pm}(\mathcal{F}^\pm) \xrightarrow{\cong} \mathcal{F}_\lambda^\pm$; $|0\rangle^\pm \mapsto |\lambda\rangle^\pm$ and a unique V^\pm -module isomorphism such that $\xi_r^\pm : (V^\pm)^r := \Delta^{r\alpha_{-1}^\pm}(V^\pm) \xrightarrow{\cong} V^\pm$; $e^{n\alpha^\pm} \mapsto e^{(n+r)\alpha^\pm}$. In this paper, we always identify $\Delta^{\pm\lambda\alpha^\pm}(\mathcal{F}^\pm)$ with \mathcal{F}_λ^\pm by the above isomorphism and simply write \mathcal{F}_λ^\pm .

B.3. Affine and $\mathcal{N} = 2$ case. Let $\theta \in \mathbb{Z}$. First, we define the Lie algebra automorphism $U_{\widehat{\mathfrak{sl}}_2}^\theta$ of $\widehat{\mathfrak{sl}}_2$ by

$$U_{\widehat{\mathfrak{sl}}_2}^\theta(H_n) = H_n + \theta K \delta_{n,0}, \quad U_{\widehat{\mathfrak{sl}}_2}^\theta(E_n) = E_{n+\theta}, \quad U_{\widehat{\mathfrak{sl}}_2}^\theta(F_n) = F_{n-\theta}, \quad U_{\widehat{\mathfrak{sl}}_2}^\theta(K) = K.$$

It is easy to verify that $U_{\widehat{\mathfrak{sl}}_2}^\theta \circ U_{\widehat{\mathfrak{sl}}_2}^{\theta'} = U_{\widehat{\mathfrak{sl}}_2}^{\theta+\theta'}$ for any $\theta, \theta' \in \mathbb{Z}$.

Next we define the Lie superalgebra automorphism $U_{\mathfrak{ns}_2}^\theta$ of \mathfrak{ns}_2 by

$$U_{\mathfrak{ns}_2}^\theta(L_n) = L_n + \theta J_n + \frac{\theta^2}{6} C \delta_{n,0}, \quad U_{\mathfrak{ns}_2}^\theta(J_n) = J_n + \frac{\theta}{3} C \delta_{n,0}, \\ U_{\mathfrak{ns}_2}^\theta(G_r^\pm) = G_{r \pm \theta}^\pm, \quad U_{\mathfrak{ns}_2}^\theta(C) = C.$$

Similarly to the affine case, we have $U_{\mathfrak{ns}_2}^\theta \circ U_{\mathfrak{ns}_2}^{\theta'} = U_{\mathfrak{ns}_2}^{\theta+\theta'}$ for any $\theta, \theta' \in \mathbb{Z}$.

Both $U_{\widehat{\mathfrak{sl}}_2}^\theta$ and $U_{\mathfrak{ns}_2}^\theta$ are called the *spectral flow automorphisms*. These automorphisms induce the endofunctors

$$\begin{aligned} (U_{\widehat{\mathfrak{sl}}_2}^\theta)^* &: \widehat{\mathfrak{sl}}_2\text{-mod} \longrightarrow \widehat{\mathfrak{sl}}_2\text{-mod}, \\ (U_{\mathfrak{ns}_2}^\theta)^* &: \mathfrak{ns}_2\text{-mod} \longrightarrow \mathfrak{ns}_2\text{-mod}. \end{aligned}$$

By some computations, we have the following.

Lemma B.4. Let $M \in \text{Obj}(\mathcal{LW}_{V_k(\widehat{\mathfrak{sl}}_2)})$ and $N \in \text{Obj}(\mathcal{LW}_{V_c(\mathfrak{ns}_2)})$. Then the identity mappings of underlying spaces

$$\text{id}_M: (U_{\widehat{\mathfrak{sl}}_2}^\theta)^*(M) \xrightarrow{\cong} M^\theta, \quad \text{id}_N: (U_{\mathfrak{ns}_2}^\theta)^*(N) \xrightarrow{\cong} N^\theta$$

give module isomorphisms. In particular, the restrictions of the functors $(U_{\widehat{\mathfrak{sl}}_2}^\theta)^*$ and $(U_{\mathfrak{ns}_2}^\theta)^*$ are naturally equivalent to $\Delta_{\widehat{\mathfrak{sl}}_2}^\theta$ and $\Delta_{\mathfrak{ns}_2}^\theta$, respectively.

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